

A direct solution to the Generic Point Problem

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Abstract

We provide a new proof of a recent theorem of Ben-Yaacov, Melleray, and Tsankov. If G is a Polish group and X is a minimal, metrizable G -flow with all orbits meager, then the universal minimal flow $M(G)$ is non-metrizable. In particular, we show that given X as above, the universal highly proximal extension of X is non-metrizable.

1 Introduction

In this paper, we are concerned with actions of a topological group G on a compact space X . All groups and spaces are assumed Hausdorff. A compact space X equipped with a continuous G -action $a : G \times X \rightarrow X$ is called a G -flow. The action a is often suppressed in the notation, i.e. gx is written for $a(g, x)$. A G -flow X is called *minimal* if every orbit is dense. It is a fact that every topological group G admits a *universal minimal flow* $M(G)$, a minimal flow which admits a G -map onto any other minimal flow. A G -map is a continuous map respecting the G -action. The flow $M(G)$ is unique up to G -flow isomorphism.

We can now recall the following theorem of Ben-Yaacov, Melleray, and Tsankov [4].

Theorem 1.1. *Let G be a Polish group, and let $M(G)$ be the universal minimal flow of G . If $M(G)$ is metrizable, then $M(G)$ has a comeager orbit.*

The question of whether or not metrizability of $M(G)$ was enough to guarantee a comeager orbit was first asked by Angel, Kechris, and Lyons [5]. In [6], the current author proved Theorem 1.1 in the case when G is the automorphism group of a first-order structure. The proof given there used topological properties of the largest G -ambit $S(G)$ along with combinatorial reasoning about the structures. In [4], the authors also use topological properties of $S(G)$, but the combinatorics is replaced by the following theorem due to Rosendal; see [4] for a proof.

Theorem 1.2. *Let G be a Polish group acting continuously on a compact metric space X . Assume the action is topologically transitive. Then the following are equivalent.*

1. G has a comeager orbit.
2. For any open $1 \in V \subseteq G$ and any open $B \subseteq X$, there is open $C \subseteq B$ so that for any open $D \subseteq C$, the set $C \setminus VD$ is nowhere dense.

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It is proven in [5] that comeager orbits push forward; namely, if X is a minimal G -flow, $x \in X$ is a point whose orbit is generic, and if $\pi : X \rightarrow Y$ is a surjective G -map, then $\pi(x)$ has generic orbit in Y . Theorem 1.1 then becomes equivalent to the following: whenever G is a Polish group and X is a minimal metrizable flow with all orbits meager, then G must admit some minimal, non-metrizable flow. Remarkably, neither [4] nor [6] prove Theorem 1.1 in this direct fashion.

We provide a direct proof of Theorem 1.1. For any topological group G and any G -flow X , we construct a new G -flow denoted $S_G(X)$. We then show that if X is minimal, then so is $S_G(X)$. Lastly, if G is Polish and X is metrizable and has all orbits meager, we use Theorem 1.2 to show that $S_G(X)$ is non-metrizable.

After providing our new proof of 1.1, we investigate the flow $S_G(X)$ in more detail. For any G -flow X , there is a natural map $\pi_X : S_G(X) \rightarrow X$. When X is minimal, we show that π_X is the *universal highly proximal extension* of X . The notion of a highly proximal extension was introduced by Auslander and Glasner in [2]. If X and Y are minimal G -flows, a G -map $\varphi : Y \rightarrow X$ is *highly proximal* if for any $x \in X$ and non-empty open $U \subseteq Y$, there is $g \in G$ with $g\pi^{-1}(\{x\}) \subseteq U$. Auslander and Glasner prove in [2] that for every minimal G -flow X , there is a *universal highly proximal extension* $\pi : \widehat{X} \rightarrow X$. This means that π is highly proximal, and for every other highly proximal $\varphi : Y \rightarrow X$, there is a G -map $\psi : \widehat{X} \rightarrow Y$ so that $\pi = \varphi \circ \psi$. The map π is unique up to G -flow isomorphism over X . Our construction of the flow $S_G(X)$ provides a new construction of the universal highly proximal extension of X and hints at a generalization of this notion even when X is not minimal.

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2 The flow $S_G(X)$ and proof of Theorem 1.1

All groups and spaces will be assumed Hausdorff. In this section, fix a topological group G and a G -flow X . Write \mathcal{N}_G for the collection of symmetric open neighborhoods of the identity in G , and write $\text{op}(X)$ for the collection of nonempty open subsets of X .

Definition 2.1. A *near filter* is any $\mathcal{F} \subseteq \text{op}(X)$ so that for any $A_1, \dots, A_k \in \mathcal{F}$ and any $U \in \mathcal{N}_G$, we have $UA_1 \cap \dots \cap UA_k \neq \emptyset$. A *near ultrafilter* is a maximal near filter.

Near ultrafilters exist by an application of Zorn's lemma. Near ultrafilters on a uniform space have been considered in [1] and [3]. Two aspects of our approach are slightly different. First, the notion of nearness is not given by the natural uniform structure on the compact Hausdorff space X . Second, instead of working with a notion of nearness on $\mathcal{P}(X)$, we are more or less working with the regular open algebra on X (see item (2) in Lemma 2.2).

Let $S_G(X)$ denote the space of near ultrafilters on $\text{op}(X)$.

Lemma 2.2.

1. Let $p \in S_G(X)$, and let $A \subseteq X$ be open. If $A \notin p$, then there is some $V \in \mathcal{N}_G$ with $VA \notin p$.
2. Let $A \subseteq X$ be open, and let $B_1, \dots, B_k \subseteq A$ be open with $B_1 \cup \dots \cup B_k$ dense in A . If $p \in S_G(X)$ and $A \in p$, then $B_i \in p$ for some $i \leq k$.

Proof.

1. As $A \notin p$, find $B_1, \dots, B_n \in p$ and $U \in \mathcal{N}_G$ with $UA \cap UB_1 \cap \dots \cap UB_n = \emptyset$. Let $V \in \mathcal{N}_G$ with $VV \subseteq U$. Then $V(VA) \cap VB_1 \cap \dots \cap VB_n = \emptyset$.
2. Towards a contradiction, assume $B_i \notin p$ for each $i \leq k$. For each $i \leq k$, find $B_1^i, \dots, B_{n_i}^i \in p$ and a $U \in \mathcal{N}_G$ so that $UB_i \cap UB_1^i \cap \dots \cap UB_{n_i}^i = \emptyset$. We can take the same $U \in \mathcal{N}_G$ for each $i \leq k$ by intersecting. Let $C = \bigcap_{i \leq k} \bigcap_{j \leq n_i} UB_j^i$. Then since $A \in p$, we have $UA \cap C \neq \emptyset$. Let $ga \in UA \cap C$, where $g \in U$ and $a \in A$. Since $UA \cap C$ is open, there is open $A' \subseteq A$ with $gA' \subseteq UA \cap C$. As $B_1 \cup \dots \cup B_k$ is dense in A , there is some $i \leq k$ and some $b \in B_i$ with $gb \in UA \cap C$. Since $gb \in UB_i$, this is a contradiction. \square

Definition 2.3. If $A \in \text{op}(X)$, set $N_A := \{p \in S_G(X) : A \notin p\}$. We endow $S_G(X)$ with the topology whose typical basic open neighborhood is N_A for $A \in \text{op}(X)$.

Proposition 2.4. *The topology from Definition 2.3 is compact Hausdorff.*

Proof. To show that $S_G(X)$ is Hausdorff, let $p \neq q \in S_G(X)$. Find some $A \in p \setminus q$. As $A \notin q$, find some $V \in \mathcal{N}_G$ so that $VA \notin q$. Set $B = \text{int}(X \setminus VA)$. Then $B \notin p$. So $p \in N_B$, $q \in N_{VA}$, and $N_{VA} \cap N_B = \emptyset$.

To show that $S_G(X)$ is compact, suppose $\mathcal{C} := \{N_{A_i} : i \in I\}$ is a collection of basic open sets without a finite subcover. Then for any $i_1, \dots, i_k \in I$, we can find $p \in \bigcap_{j \leq k} S_G(X) \setminus N_{A_{i_j}}$, equivalently, with $A_{i_1}, \dots, A_{i_k} \in p$. But this implies that $\{A_i : i \in I\}$ is a near filter, and can be extended to a near ultrafilter q . Therefore \mathcal{C} is not an open cover. \square

Definition 2.5. If $p \in S_G(X)$ and $g \in G$, we let $gp \in S_G(X)$ be defined by declaring $A \in gp$ iff $g^{-1}A \in p$ for each $A \in \text{op}(X)$.

Proposition 2.6. *The action in Definition 2.5 is continuous.*

Proof. First note that for a fixed $g \in G$, the map $p \rightarrow gp$ is continuous. So let $p_i, p \in S_G(X)$ and $g_i \in G$ with $p_i \rightarrow p$ and $g_i \rightarrow 1$. Suppose $A \notin p$. Find $V \in \mathcal{N}_G$ with $VA \notin p$. So eventually $VA \notin p_i$. Also, as $g_i \rightarrow 1$, eventually we have $g_i^{-1} \in V$. Whenever $g_i^{-1}A \subseteq VA$, we must have $g_i^{-1}A \notin p_i$. So eventually $A \notin g_i p_i$. \square

Up until now, no assumptions on G and X have been needed. In fact, we did not even need X to be compact to construct $S_G(X)$. We now begin adding extra assumptions to G and X to obtain stronger conclusions about $S_G(X)$.

Proposition 2.7. *Suppose X is a minimal G -flow. Then so is $S_G(X)$.*

Proof. Let $p \in S_G(X)$, and let $A \in \text{op}(X)$ with $N_A \neq \emptyset$. Find some $V \in \mathcal{N}_G$ with $N_{VA} \neq \emptyset$. Then $B := \text{int}(X \setminus VA) \neq \emptyset$. As X is minimal, find g_1, \dots, g_k with $X = \bigcup_{i \leq k} g_i B$. For some $i \leq k$, we must have $g_i B \in p$. Then $B \in g_i^{-1} p$, so we must have $A \notin g_i^{-1} p$, and the orbit of p is dense as desired. \square

Before proving Theorem 1.1, we need a sufficient criterion for when $S_G(X)$ is non-metrizable.

Proposition 2.8. *Suppose there are $\{A_n : n < \omega\} \subseteq \text{op}(X)$ and $V \in \mathcal{N}_G$ so that the collection $\{VA_n : n < \omega\}$ is pairwise disjoint. Then $S_G(X)$ is non-metrizable.*

Proof. If $S \subseteq \omega$, let $A_S = \bigcup_{n \in S} A_n$, and let $Y = \{p \in S_G(X) : A_\omega \in p\}$. Then $Y \subseteq S_G(X)$ is a closed subspace. To show that $S_G(X)$ is non-metrizable, we will exhibit a continuous surjection $\pi : Y \rightarrow \beta\omega$. First note that if $S \subseteq \omega$, then $VA_S \cap VA_{\omega \setminus S} = \emptyset$. Therefore, if $p \in Y$, p contains exactly one of A_S or $A_{\omega \setminus S}$ for each $S \subseteq \omega$. We let $\pi : Y \rightarrow \beta\omega$ be defined so that for $S \subseteq \omega$, $S \in \pi(p)$ iff $A_S \in p$. It is immediate that π is continuous. To see that π is surjective, let $q \in \beta\omega$. Then $\{A_S : S \in q\}$ is a near filter; any near ultrafilter p extending it is a member of Y with $\pi(p) = q$. \square

Proof of Theorem 1.1. We now fix a Polish group G and a minimal G -flow X whose orbits are all meager. Then by Theorem 1.2, there is $U \in \mathcal{N}_G$ and open $B \subseteq X$ so that for any open $C \subseteq B$, there is open $D \subseteq C$ with $C \setminus UD$ somewhere dense (since C and UD are open, this is the same as $C \setminus UD$ having nonempty interior).

Let $V \in \mathcal{N}_G$ with $VV \subseteq U$. We now produce $\{A_n : n < \omega\} \subseteq \text{op}(X)$ with $\{VA_n : n < \omega\}$ pairwise disjoint. First set $B_0 = B$. As $B_0 \subseteq B$, there is $A_0 \subseteq B_0$ so that $B_0 \setminus UA_0$ has nonempty interior. Suppose open sets B_0, \dots, B_{n-1} and A_0, \dots, A_{n-1} have been produced so that $A_i \subseteq B_i$ and $\text{int}(B_i \setminus UA_i) \neq \emptyset$. We continue by setting $B_n = \text{int}(B_{n-1} \setminus UA_{n-1})$. As $B_n \subseteq B$, there is $A_n \subseteq B_n$ so that $B_n \setminus UA_n$ has nonempty interior. Notice that for any $m \leq n$, we also have $A_n \subseteq B_m$. It follows that if $m < n$, we have $UA_m \cap A_n = \emptyset$. This implies that $VA_m \cap VA_n = \emptyset$ as desired. We can now apply Proposition 2.8 to conclude that $S_G(X)$ is not metrizable. \square

3 Universal highly proximal extensions

Let $\varphi : Y \rightarrow X$ be a G -map between minimal flows. There are several equivalent definitions which all say that φ is *highly proximal*. The definition we will use here is that φ is highly proximal iff every non-empty open $B \subseteq Y$ contains a fiber $\varphi^{-1}(\{x\})$ for some $x \in X$. Define the *fiber image* of B to be the set $\varphi_{fib}(B) := \{x \in X : \varphi^{-1}(\{x\}) \subseteq B\}$. Notice that $\varphi_{fib}(B)$ is open, and φ is highly proximal iff $\varphi_{fib}(B) \neq \emptyset$ for every non-empty open $B \subseteq Y$. It follows that this definition is the same as the one given in the introduction.

Now let X be a G -flow, and form $S_G(X)$. We define the map $\pi_X : S_G(X) \rightarrow X$ as follows. For each $p \in S_G(X)$, there is a unique $x_p \in X$ so that every neighborhood of x_p is in p . The existence of such a point is an easy consequence of the compactness of X and the second item of 2.2. For uniqueness, notice that if $x \neq y \in X$, we can find open

$A \ni x$, $B \ni y$ and $U \in \mathcal{N}_G$ with $UA \cap UB = \emptyset$. We set $\pi_X(p) = x_p$. This map clearly respects the G -action. To check continuity, one can check that if $K \subseteq X$ is closed, then $\pi_X^{-1}(K) = \{p \in S_G(X) : A \in p \text{ for every open } A \supseteq K\}$, and this is a closed condition.

Proposition 3.1. *Let X be minimal. Then the map $\pi_X : S_G(X) \rightarrow X$ is highly proximal.*

Proof. By 2.7, $S_G(X)$ is a minimal flow. So let $N_A \subseteq S_G(X)$ be a nonempty basic open neighborhood. This implies that $\text{int}(X \setminus A) \neq \emptyset$. Let $x \in \text{int}(X \setminus A)$. Then there are open $B \ni x$ and $U \in \mathcal{N}_G$ with $UB \cap A = \emptyset$. It follows that any $p \in S_G(X)$ containing B cannot contain A . In particular, we have $\pi_X^{-1}(\{x\}) \subseteq N_A$. \square

Theorem 3.2. *Let X be minimal. Then the map $\pi_X : S_G(X) \rightarrow X$ is the universal highly proximal extension of X .*

Proof. Fix a highly proximal extension $\varphi : Y \rightarrow X$. For each $y \in Y$, let $\mathcal{F}_y := \{\varphi_{fib}(B) : B \ni y \text{ open}\}$. Then $\mathcal{F}_y \subseteq \text{op}(X)$ is a filter of open sets, so in particular it is a near filter. We will show that for each $p \in S_G(X)$, there is a unique $y \in Y$ with $\mathcal{F}_y \subseteq p$. This will define the map $\psi : S_G(X) \rightarrow Y$.

We first show that for each $p \in S_G(X)$, there is at least one such $y \in Y$. To the contrary, suppose for each $y \in Y$, there were $B_y \ni y$ open so that $\varphi_{fib}(B_y) \notin p$. Find y_1, \dots, y_k so that $\{B_{y_1}, \dots, B_{y_k}\}$ is a finite subcover. Let $A_i = \varphi_{fib}(B_{y_i})$. Each A_i is open, so we will reach a contradiction once we show that $\bigcup_{i \leq k} A_i$ is dense. Let $A \subseteq X$ be open. Then $C := B_{y_i} \cap \varphi^{-1}(A) \neq \emptyset$ for some $i \leq k$. As C is open, $\varphi_{fib}(C) \neq \emptyset$, and $\varphi_{fib}(C) \subseteq A \cap A_i$.

Now we consider uniqueness. Let $p \in S_G(X)$, and consider $y \neq z \in Y$. Find open $B \ni y$ and $C \ni z$ and some $V \in \mathcal{N}_G$ so that $VB \cap VC = \emptyset$. It follows that $\varphi_{fib}(VB) \cap \varphi_{fib}(VC) = \emptyset$. Now notice that $V\varphi_{fib}(B) \subseteq \varphi_{fib}(VB)$, and likewise for C . Hence p cannot contain both \mathcal{F}_y and \mathcal{F}_z .

The map ψ clearly respects the G -action and satisfies $\pi_X = \varphi \circ \psi$. To show continuity, let $K \subseteq Y$ be closed. Let $\mathcal{F}_K := \{\varphi_{fib}(B) : B \supseteq K \text{ open}\}$. We will show that $\psi(p) \in K$ iff $\mathcal{F}_K \subseteq p$. From this it follows that $\psi^{-1}(K)$ is closed. One direction is clear. For the other, suppose $\psi(p) = y \notin K$. Find open sets $B \ni y$, $C \supseteq K$, and $V \in \mathcal{N}_G$ with $VB \cap VC = \emptyset$. As in the proof of uniqueness, p cannot contain both \mathcal{F}_y and \mathcal{F}_K . \square

By combining the main results of the previous two sections, we obtain the following.

Corollary 3.3. *Let G be a Polish group, and let X be a minimal, metrizable G -flow with all orbits meager. Then the universal highly proximal extension of X is non-metrizable.*

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